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EXTENSIONS OF SEMIGROUPS AND MORPHISMS OF SEMIGROUP C^*-ALGEBRAS

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Abstract: The paper is devoted to the normal extensions of discrete semigroups and $*$-homomorphisms of semigroup C^*-algebras. We study the normal extensions of abelian semigroups by arbitrary groups. Considering numerical semigroups, we prove that they are normal extensions of the semigroup of nonnegative integers by finite cyclic groups. On the other hand, we prove that if a semigroup is a normal extension of the semigroup of nonnegative integers by a finite cyclic group generated by a single element then this semigroup is isomorphic to a numerical semigroup. As regard a normal extension with a generating set, we consider two reduced semigroup C^*-algebras defined by this extension. We show that there exists an embedding of the semigroup C^*-algebras which is generated by an injective homomorphism of the semigroups and the natural isometric representations of these semigroups.

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Introduction

The article is devoted to studying the normal extensions of abelian cancellative semigroups and the embeddings of reduced semigroup C^*-algebras which are generated by the extensions.

One of the sources of motivation of the present article is the results of the theory of operator algebras which concern the embeddings of semigroup C^*-algebras. These results were first obtained by Coburn [1, 2] and Douglas [3] with the aim of their application to the theory of operator index. These authors considered the reduced semigroup C^*-algebras $C_r^*(\Gamma^+)$ for the semigroups Γ^+ that are the positive cones of ordered semigroups Γ in the additive group of the reals. They showed in particular that, for every isometric representation of the positive cone Γ^+ by nonunitary elements in an arbitrary unital C^*-algebra A, there exists a unique embedding of the algebra $C_r^*(\Gamma^+)$ into A. For the case when Γ^+ is the semigroup of nonnegative integers $\mathbb{Z}^+$, this result is known in the theory of operator algebras as Coburn's Theorem [4, Theorem 3.5.18]. This theorem was generalized in [5, 6] by Murphy to the case of positive cones in ordered groups and was further developed in the works by Nica [7] as well as by Laca and Raeburn [8].

In turn, considering the numerical semigroup $\mathbb{Z}^+ \setminus \{1\}$, Murphy [5] and Jang [9, 10] proved that the analogous result does not hold for the embeddings of C^*-algebras. This fact initiated the study of the properties of isometric representations and the structures of semigroup C^*-algebras for the whole class of the so-called numerical semigroups whose example is given by $\mathbb{Z}^+ \setminus \{1\}$ (see, for instance, [11–13]).

Another source of motivation for this article is the theory of semigroup extensions. As is known, extensions play an important role in the study of the structure and the characteristics of semigroups; in particular, their cohomology (see, for instance, [14]). Recall that there are various kinds of extensions under study semigroups. For instance, one of the first works [15] by Clifford on this topic is devoted to ideal extensions of semigroups. In [16], Rédei introduced and studied Schreier extensions. In [17, 18], Gluskin and Perepelitsyn studied normal extensions of semigroups.

The present article is a continuation of the study of the properties of reduced semigroup C^*-algebras and their involutive homomorphisms which was initiated in [19–29]. It should be noted however that the

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aim of the article is twofold: On the one hand, the article deals with the normal extensions of semigroups by arbitrary groups. Moreover, some results are obtained on the properties of extensions of $\mathbb{Z}^+$. It is proved in particular that if a semigroup L is a normal extension of $\mathbb{Z}^+$ by a finite cyclic group generated by a single element then L is isomorphic to a numerical semigroup. Conversely, we show that each numerical semigroup can act as a normal extension of $\mathbb{Z}^+$ by a finite cyclic group. On the other hand, for a normal extension (L, τ, σ) of a semigroup S by an arbitrary group, we consider the question of existence of an embedding of the semigroup C^*-algebra $C_r^*(S)$ into $C_r^*(L)$ which is generated by an injective homomorphism $\tau : S \longrightarrow L$ of abelian semigroups and the natural isometric representations of S and L in the corresponding semigroup C^*-algebras. In the article, we answer the question in the affirmative for some class of normal extensions. Namely, we prove a theorem on the embeddings of the semigroup C^*-algebras in the case when the extension (L, τ, σ) admits a generating set. The theorem generalizes the assertion of [29] for one particular case of an extension of a semigroup S which is defined by means of a finite cyclic group generated by exactly one element of a semigroup L. Observe for completeness that if the extension (L, τ, σ) does not admit a generating set then the above-mentioned embedding of the semigroup C^*-algebras may fail to exist. We illustrate this by an example of a normal extension of semigroups.

Note that the results of the article are related to the functoriality question for morphisms of semigroup C^*-algebras which was posed in [30]. Namely, let $C^*(P)$ denote the universal C^*-algebra of a semigroup P which is generated by isometries $\{v_p, p \in P\}$. Does a homomorphism $\varphi : P \longrightarrow Q$ of cancellative semigroups induce a morphism of the corresponding semigroup C^*-algebras $C^*(P) \longrightarrow C^*(Q)$ by the formula $v_p \mapsto v_{\varphi(p)}$, $p \in P$?

The article is organized as follows: It consists of an introduction and three sections. Section 1 contains the necessary information from the theory of semigroup extensions and provides the notion of a normal extension of a semigroup by a group that admits a generating set. Considering the extensions generated by a single element, we prove a criterion for the equivalence of normal extensions. In Section 2, we consider the extensions of $\mathbb{Z}^+$ by numerical semigroups with the aid of finite cyclic groups. Section 3 is devoted to the reduced semigroup C^*-algebras for two semigroups one of which is a normal extension of the other by means of an arbitrary group. We prove a theorem on the embedding of these semigroup C^*-algebras.

1. Extensions of Semigroups by Groups

Throughout the article, we let S and L designate the discrete additive cancellative semigroups with neutral elements. Denote an arbitrary abelian group by Γ. The neutral elements of S, L, and Γ will be denoted by 0.

The definition of a normal extension of a semigroup is contained in [17, 31]. Let us give the definition of a normal extension of a semigroup by a group.

Suppose that we have an injective semigroup homomorphism $\tau : S \longrightarrow L$ and a surjective semigroup homomorphism $\sigma : L \longrightarrow \Gamma$. Refer to the triple (L, τ, σ) as a *normal extension* of S by Γ if $\tau(S)$ is the full preimage of the neutral element of Γ; i.e., $\sigma^{-1}(0) = \tau(S)$.

Thus, a normal extension can be defined with the use of a short exact sequence

$$0 \longrightarrow S \xrightarrow{\tau} L \xrightarrow{\sigma} \Gamma \longrightarrow 0.$$

In this case, the semigroup L is also called an *extension of S by* Γ.

Two extensions (L, τ, σ) and (L', τ', σ') of S by Γ are called *equivalent* if there is a semigroup isomorphism $\psi : L \longrightarrow L'$ making the diagram

$$\begin{tikzcd} & L \arrow[dd, "\psi"] \arrow[dr, "\sigma"] & \\ S \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & & \Gamma \\ & L' \arrow[ur, "\sigma'"'] & \end{tikzcd} \tag{1}$$

commute.

Obviously, if L is a normal extension by Γ then L is representable as the disjoint union

$$L = \bigsqcup_{\gamma\in\Gamma} L_\gamma, \tag{2}$$

where $L_\gamma = \sigma^{-1}(\gamma)$. Note that $L_0 = \tau(S)$.

Let a set X be such that $X \subset L \setminus \tau(S)$ and $X \cap L_\gamma = \{x_\gamma\}$ for every $\gamma \in \Gamma$, $\gamma \neq 0$. We say that the extension (L, τ, σ) of S is *generated by* X if each $y \in L \setminus \tau(S)$ is uniquely representable as $y = \tau(a) + x_\gamma$ for some $a \in S$ and $\gamma \in \Gamma$. In this case every subset L_γ, $\gamma \neq 0$, has the form

$$L_\gamma = \tau(S) + x_\gamma := \{\tau(a) + x_\gamma \mid a \in S\},$$

and L is representable as the disjoint union

$$L = \tau(S) \sqcup \Big(\bigsqcup_{x_\gamma \in X} (\tau(S) + x_\gamma) \Big). \tag{3}$$

If X is finite then we refer to the extension (L, τ, σ) as a *finitely generated* normal extension. In this case, Γ is finite.

Distinguish one more class of extensions. Let Γ be a finite cyclic group. Then, up to isomorphism, $\Gamma = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the residue class group modulo n. Let a triple (L, τ, σ) be a normal extension of S by $\mathbb{Z}_n$. If there exists $x \in L \setminus \tau(S)$ such that every $y \in L$ is uniquely representable as

$$y = \tau(a) + kx \tag{4}$$

for some $a \in S$ and $k \in \mathbb{Z}^+$, $0 \leq k \leq n-1$, then we refer to the triple (L, τ, σ) as a normal extension of S *generated by an element* x. In this case, L is representable as the disjoint union

$$L = \bigsqcup_{k=0}^{n-1} (\tau(S) + kx). \tag{5}$$

Obviously, a normal extension by $\mathbb{Z}_n$ generated by a single element is a particular case of a finitely generated normal extension.

Note that the above extensions with generating sets are Schreier extensions (see [14, 16]).

Henceforth, we denote the elements of $\mathbb{Z}_n$ by $[0]_n, \ldots, [n-1]_n$.

In [29], Gumerov considered the normal extension (L_x, τ, σ) by $\mathbb{Z}_n$ generated by an element x such that $\sigma(x) = [m]_n$, where the numbers m and n are coprime. In the following lemma, we prove that this property is in fact possessed by every normal extension of the type.

Lemma 1.1. *Let (L, τ, σ) be a normal extension of S by $\mathbb{Z}_n$ which is generated by an element x. Then the following hold:*

(1) *$\sigma(x) = [m]_n$, where m and n are coprime;*

(2) *$nx \in \tau(S)$.*

Proof. (1): Let $\sigma(x) = [m]_n$, where m and n are not coprime. Since σ is a surjection, there is $y \in L$ such that $\sigma(y) = [1]_n$. By the definition of a normal extension generated by x, we see that $y = \tau(a) + kx$ for some $a \in S$ and $k \in \mathbb{Z}^+$, $0 \le k \le n-1$. Then $\sigma(kx) = [1]_n$. On the other hand, $\sigma(kx) = [km]_n$. Thus, $km - nl = 1$ for some $l \in \mathbb{N}$, which contradicts the fact that m and n have a common divisor different from unity.

(2): Indeed, since $\sigma(nx) = [nm]_n = [0]_n$ and $\sigma^{-1}([0]_n) = \tau(S)$; therefore, $nx \in \tau(S)$. $\square$

The following assertion contains sufficient conditions for the two extensions generated by one element to equivalent.

Proposition 1.1. *Let (L, τ, σ) and (L', τ', σ') be two normal extensions of S by $\mathbb{Z}_n$ which are generated by elements x and x' such that $\sigma(x) = \sigma'(x')$. Let $\tau^{-1}(nx) = (\tau')^{-1}(nx')$. Then the extensions (L, τ, σ) and (L', τ', σ') are equivalent.*

Proof. Construct some semigroup isomorphism $\psi : L \longrightarrow L'$ such that diagram (1) commutes. By the definition of a normal extension generated by a single element, every element $y \in L$ is uniquely representable in the form (4). Put $\psi(y) = \tau'(a) + kx'$. Owing to the existence of $b \in S$ such that $nx = \tau(b)$ and $nx' = \tau'(b)$, we see that $\psi(nx) = \psi(\tau(b)) = \tau'(b) = nx'$. Therefore, ψ is a semigroup homomorphism. The uniqueness of the corresponding representations (4) of the elements of L and L' implies that ψ is an isomorphism.

Check that diagram (1) commutes. The equality $\psi \circ \tau = \tau'$ is obvious. Verify that $\sigma' \circ \psi = \sigma$. Indeed, since $\sigma(x) = \sigma'(x')$; therefore, for every $y \in L$ from (4) we infer that

$$(\sigma' \circ \psi)(y) = (\sigma' \circ \psi)(\tau(a) + kx) = \sigma'(\tau'(a) + kx') = k\sigma'(x') = k\sigma(x) = \sigma(y). \quad \square$$

The converse to Proposition 1.1 fails in general. But if there is no nontrivial subgroup of S then we have

Proposition 1.2. *Let a semigroup S be such that its every semigroup is isomorphic to the trivial group. Let (L, τ, σ) and (L', τ', σ') be the two equivalent normal extensions of S by $\mathbb{Z}_n$ which are generated respectively by elements x and x' such that $\sigma(x) = \sigma'(x')$. Then $(\tau)^{-1}(nx) = (\tau')^{-1}(nx')$.*

Proof. Suppose that there exists a semigroup isomorphism $\psi : L \longrightarrow L'$ making diagram (1) commute. Then $\psi \circ \tau = \tau'$, i.e., $\psi(\tau(a)) = \tau'(a)$ for every $a \in S$. Show that $\psi(x) = x'$. Suppose that $\psi(x) = \tau'(c) + kx'$ for some $c \in S$ and $0 \le k \le n-1$. Since diagram (1) commutes, we infer that $\sigma'(\psi(x)) = \sigma(x) = [m]_n$, where $m \in \mathbb{N}$, $1 \le m \le n-1$. On the other hand, $\sigma'(\psi(x)) = \sigma'(\tau'(c) + kx') = [km]_n$. Thus, $[m]_n = [km]_n$ or $(k-1)m = ln$ for some $l \in \mathbb{N}$. By Lemma 1.1, m and n are coprime. Therefore, reckoning with the condition $0 \le k \le n-1$, we see that $[m]_n = [km]_n$ holds only if $k = 1$. Consequently, $\psi(x) = \tau'(c) + x'$. We can prove similarly that $\psi^{-1}(x') = \tau(d) + x$ for some $d \in S$. Then

$$x' = \psi(\tau(d) + x) = \tau'(d) + \tau'(c) + x'.$$

So, $\tau'(d) + \tau'(c) = 0$ and $d + c = 0$. This gives that $d = c = 0$. Otherwise, S has a nontrivial subgroup. Thus, $\psi(x) = x'$. It remains to observe that if $nx = \tau(b)$ and $nx' = \tau'(b')$ for $b, b' \in S$ then $\tau'(b) = \psi(\tau(b)) = \psi(nx) = nx' = \tau'(b')$. Hence, $b = b'$ by the injectivity of τ. $\square$

Closing this section, we give the example showing that we cannot omit in Proposition 1.2 the requirement that S must have no nontrivial subgroup.

Example 1.1. Let $L = \mathbb{Z}_3 \times \mathbb{Z}_2$. Define the injective group homomorphisms

$$\tau : \mathbb{Z}_3 \longrightarrow L : [k]_3 \mapsto ([k]_3, [0]_2), \quad \tau' : \mathbb{Z}_3 \longrightarrow L : [k]_3 \mapsto ([3-k]_3, [0]_2)$$

and the surjective group homomorphism

$$\sigma : L \longrightarrow \mathbb{Z}_2 : ([k]_3, [l]_2) \mapsto [l]_2.$$

Then (L, τ, σ) and (L, τ', σ) are the two normal extensions of $\mathbb{Z}_3$ by $\mathbb{Z}_2$ which are generated by $x = ([1]_3, [1]_2)$. If we consider the isomorphism $\psi : L \longrightarrow L$ defined by $\psi([k]_3, [l]_2) = ([3-k]_3, [l]_2)$ then diagram (1) commutes, and so the extensions (L, τ, σ) and (L, τ', σ) are equivalent. Moreover, $(\tau)^{-1}(2x) \ne (\tau')^{-1}(2x)$.

2. Extensions and Numerical Semigroups

In this section, we consider as S the set of nonnegative integers $\mathbb{Z}^+$ with the natural addition. To formulate further results, we recall the definition of a numerical semigroup. The facts of the theory of numerical semigroups can be found, for instance, in [32].

A *numerical semigroup* is a nontrivial subsemigroup of $\mathbb{Z}^+$ containing 0 and such that the greatest common divisor of its elements is 1.

We can give the equivalent definition of a numerical semigroup as a subsemigroup of $\mathbb{Z}^+$ whose complement in $\mathbb{Z}^+$ is finite. The following result is proved in [32]: Each nontrivial semigroup with zero in $\mathbb{Z}^+$ is a numerical semigroup up to isomorphism.

It is known that every numerical semigroup has finitely many generators. Following [32], let $Z^{n_1,\dots,n_p}$ stand for the numerical semigroup that is generated by $\{n_1,\dots,n_p\}$. Thus,

$$Z^{n_1,\dots,n_p} = \left\{ \sum_{i=1}^{p} a_i n_i \mid a_i \in \mathbb{Z}^+ \right\}.$$

Note that the representation of an element of a numerical semigroup $Z^{n_1,\dots,n_p}$ as a sum $\sum_{i=1}^{p} a_i n_i$ is nonunique.

In the following assertion, we prove that each numerical semigroup $Z^{n_1,\dots,n_p}$ can act as a normal extension of $\mathbb{Z}^+$ by $\mathbb{Z}_n$, where n is an arbitrary element of $Z^{n_1,\dots,n_p}$.

Proposition 2.1. *For every numerical semigroup $Z^{n_1,\dots,n_p}$ and each $n \in Z^{n_1,\dots,n_p}$, there exist an injective semigroup homomorphism $\tau : \mathbb{Z}^+ \longrightarrow Z^{n_1,\dots,n_p}$ and a surjective semigroup homomorphism $\sigma : Z^{n_1,\dots,n_p} \longrightarrow \mathbb{Z}_n$ such that the short sequence*

$$0 \longrightarrow \mathbb{Z}^+ \xrightarrow{\ \tau\ } Z^{n_1,\dots,n_p} \xrightarrow{\ \sigma\ } \mathbb{Z}_n \longrightarrow 0$$

is exact.

Proof. Define the homomorphisms $\tau : \mathbb{Z}^+ \longrightarrow Z^{n_1,\dots,n_p}$ and $\sigma : Z^{n_1,\dots,n_p} \longrightarrow \mathbb{Z}_n$ as follows:

$$\tau(k) = nk, \quad \sigma(z) = [z]_n$$

for all $k \in \mathbb{Z}^+$ and $z \in Z^{n_1,\dots,n_p}$. Obviously, τ is injective and σ is surjective. It is not hard to see that $\sigma^{-1}([0]_n) = \tau(\mathbb{Z}^+)$. Therefore, the triple $(Z^{n_1,\dots,n_p}, \tau, \sigma)$ is a normal extension of $\mathbb{Z}^+$ by $\mathbb{Z}_n$. $\square$

Henceforth, we consider numerical semigroups with two generators. The following proposition contains a necessary condition for such a numerical semigroup to be a normal extension of $\mathbb{Z}^+$ which is generated by a single element.

Proposition 2.2. *Suppose that a triple $(Z^{n_1,n_2}\tau, \sigma)$ is a normal extension of $\mathbb{Z}^+$ by $\mathbb{Z}_n$ which is generated by a single element. Then either $n = n_1$ or $n = n_2$.*

Proof. Let x be a generating element of the extension $(Z^{n_1,n_2}, \tau, \sigma)$. Then every element $z \in Z^{n_1,n_2}$ is uniquely representable as

$$z = \tau(k) + lx = k\tau(1) + lx,$$

where $k, l \in \mathbb{Z}^+$, $0 \le l \le n-1$. Since $\tau(1), x \in Z^{n_1,n_2}$, we have

$$\tau(1) = a_1 n_1 + b_1 n_2, \ x = a_2 n_1 + b_2 n_2$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}^+$. Thus, $n_1 = ka_1n_1 + la_2n_1 + kb_1n_2 + lb_2n_2$ for some k and l. This gives the two systems of equalities: $ka_1 = 1$, $la_2 = 0$, $kb_1 = 0$, $lb_2 = 0$ or $ka_1 = 0$, $la_2 = 1$, $kb_1 = 0$, $lb_2 = 0$. From these equalities it is easy to conclude that either $\tau(1) = n_1$ or $x = n_1$. Taking n_2 instead of n_1, we conclude that either $\tau(1) = n_2$ or $x = n_2$. Thus, each $z \in Z^{n_1,n_2}$ is uniquely representable either as $z = kn_1 + ln_2$ or $z = kn_2 + ln_1$, where $k, l \in \mathbb{Z}^+$, $0 \le l \le n-1$. In the first case, we obtain an extension by $\mathbb{Z}_{n_1}$; and in the second, by $\mathbb{Z}_{n_2}$. $\square$

In the following examples, we represent the semigroup $Z^{2,3} = \mathbb{Z}^+ \setminus \{1\}$ as an extension of $\mathbb{Z}^+$.

EXAMPLE 2.1. Let $n = 4$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}^+ \xrightarrow{\tau} \mathbb{Z}^+ \setminus \{1\} \xrightarrow{\sigma} \mathbb{Z}_4 \longrightarrow 0,$$

where $\tau(k) = 4k$ and $\sigma(m) = [m]_4$ for all $k \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+ \setminus \{1\}$. Then $\mathbb{Z}^+ \setminus \{1\}$ is representable in the form (3):

$$\mathbb{Z}^+ \setminus \{1\} = \tau(\mathbb{Z}^+) \sqcup (\tau(\mathbb{Z}^+) + 2) \sqcup (\tau(\mathbb{Z}^+) + 3) \sqcup (\tau(\mathbb{Z}^+) + 5).$$

The extension is finitely generated.

EXAMPLE 2.2. Let $n = 3$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}^+ \xrightarrow{\tau} \mathbb{Z}^+ \setminus \{1\} \xrightarrow{\sigma} \mathbb{Z}_3 \longrightarrow 0,$$

where $\tau(k) = 3k$ and $\sigma(m) = [m]_3$ for all $k \in \mathbb{Z}^+$, and $m \in \mathbb{Z}^+ \setminus \{1\}$. In this case, the semigroup $\mathbb{Z}^+ \setminus \{1\}$ is representable in the form (5):

$$\mathbb{Z}^+ \setminus \{1\} = \tau(\mathbb{Z}^+) \sqcup (\tau(\mathbb{Z}^+) + 2) \sqcup (\tau(\mathbb{Z}^+) + 2 \cdot 2).$$

The extension is generated by a single element $x = 2$.

Example 2.2 is an illustration of the following assertion:

Proposition 2.3. *Suppose that a triple $(Z^{n_1,n_2}, \tau, \sigma)$ is a normal extension of $\mathbb{Z}^+$ by $\mathbb{Z}_{n_1}$, where*

$$\tau(k) = kn_1, \quad \sigma(z) = [z]_{n_1}$$

for all $k \in \mathbb{Z}^+$ and $z \in Z^{n_1,n_2}$. Then $(Z^{n_1,n_2}, \tau, \sigma)$ is an extension generated by n_2.

PROOF. Show that each $z \in Z^{n_1,n_2}$ is uniquely representable as $z = kn_1 + ln_2$, where $k, l \in \mathbb{Z}^+$, $0 \leq l \leq n_1 - 1$. Indeed, since $z = sn_1 + tn_2$ for some $s, t \in \mathbb{Z}^+$, there obviously exist $d, l \in \mathbb{Z}^+$, $0 \leq l \leq n_1 - 1$, such that $z = (s + dn_2)n_1 + ln_2 = kn_1 + ln_2$. Prove that this representation is unique. Let $k_1n_1 + l_1n_2 = k_2n_1 + l_2n_2$ for some $k_1, k_2, l_1, l_2 \in \mathbb{Z}^+$, $0 \leq l_1, l_2 \leq n_1 - 1$. If $l_1 = l_2$ then $k_1 = k_2$. Suppose, for instance, that $l_2 > l_1$. Then $(k_1 - k_2)n_1 = (l_2 - l_1)n_2$. But this contradicts the fact that n_1 and n_2 are coprime and $l_2 - l_1 < n_1$. Thus, we have a unique representation of an arbitrary $z \in Z^{n_1,n_2}$ in the form $z = kn_1 + ln_2 = \tau(k) + ln_2$, where $k, l \in \mathbb{Z}^+$, $0 \leq l \leq n_1 - 1$. $\square$

In [28], we proposed a construction that enables us to build the normal extensions of semigroups by $\mathbb{Z}_n$ which are generated by a single element and gave an example of such an extension for the semigroup of $\mathbb{Z}^+$. Namely, given $n, m \in \mathbb{N} \setminus \{1\}$, we constructed the normal extension $(L_{n,m}, \tau, \sigma)$ of $\mathbb{Z}^+$ by $\mathbb{Z}_n$ which is generated by a single element and such that the equation $nx = \tau(m)$ is solvable in $L_{n,m}$. It has been announced that $L_{n,m}$ is isomorphic to a numerical semigroup if and only if the numbers n and m are coprime.

The following theorem is a generalization of the above assertion:

Theorem 2.1. *Suppose that a triple (L, τ, σ) is a normal extension of $\mathbb{Z}^+$ by $\mathbb{Z}_n$ which is generated by an element x. Let $m \in \mathbb{N} \setminus \{1\}$ be such that*

$$nx = \tau(m), \quad \sigma(x) = [m]_n$$

and let n and m be coprime. Then (L, τ, σ) is equivalent to the extension $(Z^{n,m}, \tau', \sigma')$ of $\mathbb{Z}^+$ by $\mathbb{Z}_n$ such that

$$\tau'(k) = kn, \quad \sigma'(z) = [z]_n,$$

where $k \in \mathbb{Z}^+$ and $z \in Z^{n,m}$.

PROOF. By Proposition 2.3, $(Z^{n,m}, \tau', \sigma')$ is an extension generated by m. Since

$$\sigma(x) = [m]_n = \sigma'(m), \quad \tau^{-1}(nx) = \{m\} = (\tau')^{-1}(nm),$$

the extensions (L, τ, σ) and $(Z^{n,m}, \tau', \sigma')$ satisfy the hypotheses of Proposition 1.1. Therefore, they are equivalent. $\square$

3. Extensions and Semigroup C^*-Algebras

Let us consider an arbitrary cancellative semigroup P. We write the semigroup operation additively. Introduce the Hilbert space on P. This is the space $l^2(P)$ of square-integrable complex-valued functions on P. Denote by e_p, $p \in P$, the element of $l^2(P)$ which is defined as

$$e_p(q) := \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{if } p \neq q, \end{cases}$$

where $q \in P$. Then $\{e_p \mid p \in P\}$ is an orthonormal basis for $l^2(P)$.

In the algebra of all bounded linear operators $B(l^2(P))$ on $l^2(P)$, consider the C^*-subalgebra $C_r^*(P)$ generated by the set of isometries $\{T_p \mid p \in P\}$, where $T_p(e_q) = e_{p+q}$, $p, q \in P$. Note that $C_r^*(P)$ is called the *reduced semigroup C^*-algebra*. The identity element of $C_r^*(P)$ will be denoted by I.

Recall the definition of an isometric representation of a semigroup [5]. Let $\mathfrak{A}$ be a unital C^*-algebra. A mapping $\rho : P \longrightarrow \mathfrak{A}$ is called an *isometric representation* if $\rho(p)^*\rho(p) = I$ and $\rho(p+q) = \rho(p)\rho(q)$ for all $p, q \in P$. Obviously,

$$\pi_P : P \longrightarrow C_r^*(P) : p \mapsto T_p \tag{6}$$

is an isometric representation. We will refer to the representation defined by (6) as the *natural representation* of P in $C_r^*(P)$.

Let (L, τ, σ) be a normal extension of S by Γ. Consider the C^*-algebras $C_r^*(S)$ and $C_r^*(L)$ defined by $\{T_a \mid a \in S\}$ and $\{T_y \mid y \in L\}$ respectively.

In the C^*-algebra $C_r^*(S)$, consider all possible products of the operators T_a and T_a^* of the form

$$V = T_{a_n}^{i_n} T_{a_{n-1}}^{i_{n-1}} \dots T_{a_1}^{i_1}, \tag{7}$$

where $a_1, \dots, a_n \in S$, $i_1, \dots, i_n \in \{0, 1\}$, and $T_{a_j}^0 := T_{a_j}$, $T_{a_j}^1 := T_{a_j}^*$.

The set of all operators of the form (7) constitutes an involutive semigroup which will be denoted by Mon.

All finite linear combinations of operators of the form (7)

$$A = \sum_{i=1}^{m} \alpha_i V_i \tag{8}$$

constitute a dense involutive subalgebra of $C_r^*(S)$ which we will denote by $P(S)$.

In the C^*-algebra $C_r^*(L)$, consider the subsemigroup Mon_τ whose elements have the form

$$V_\tau = T_{\tau(a_n)}^{i_n} T_{\tau(a_{n-1})}^{i_{n-1}} \dots T_{\tau(a_1)}^{i_1}, \tag{9}$$

where $a_1, \dots, a_n \in S$, $i_1, \dots, i_n \in \{0, 1\}$, and $T_{\tau(a_j)}^0 := T_{\tau(a_j)}$, $T_{\tau(a_j)}^1 := T_{\tau(a_j)}^*$. The dense involutive C^*-algebra $C_r^*(L)$ contains the subalgebra $P(\tau(S))$ whose every element is representable as

$$A_\tau = \sum_{i=1}^{m} \alpha_i (V_i)_\tau. \tag{10}$$

Owing to the validity of decomposition (2), $l^2(L)$ is representable as the direct sum of subspaces

$$l^2(L) = \bigoplus_{\gamma \in \Gamma} H_\gamma, \tag{11}$$

where $\{e_y \mid y \in L_\gamma\}$ is an orthonormal basis for H_γ.

Lemma 3.1. *For every $\gamma \in \Gamma$ and every $a \in S$, the subspace H_γ is invariant under $T_{\tau(a)}$, and so H_γ is invariant under every $V_\tau \in \mathrm{Mon}_\tau$ and every linear combination $A_\tau \in P(\tau(S))$.*

PROOF. Indeed, let $\gamma \neq 0$. Calculate $T_{\tau(a)}$ at the basis vectors $e_y \in H_\gamma$. We obtain $T_{\tau(a)}e_y = e_{\tau(a)+y} \in H_\gamma$ since $\sigma(\tau(a)+y) = \sigma(y) = \gamma$. $\square$

The following result was announced in [28]: Let (L, τ, σ) be a normal extension of S by $\mathbb{Z}_n$ which is generated by a single element. Then there is an embedding of semigroup C^*-algebras $\varphi : C_r^*(S) \longrightarrow C_r^*(L)$ generated by an isometric representation of S. The proof of this theorem is in [29]. Below we prove a generalization of this theorem for a normal extension of a semigroup by a group admitting an arbitrary generating set.

Theorem 3.1. *Let (L, τ, σ) be a normal extension of S by a group Γ with a generating set. Let $\pi_S : S \longrightarrow C_r^*(S)$ and $\pi_L : L \longrightarrow C_r^*(L)$ be natural isometric representations of S and L. Then there exists a unique unital isometric $*$-homomorphism $\varphi : C_r^*(S) \longrightarrow C_r^*(L)$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "\tau"] \arrow[d, "\pi_S"'] & L \arrow[d, "\pi_L"] \\ C_r^*(S) \arrow[r, "\varphi"'] & C_r^*(L) \end{tikzcd}$$

commutes; i.e., $\varphi \circ \pi_S = \pi_L \circ \tau$.

PROOF. We must show that the mapping φ, defined on the generators of the C^*-algebra $C_r^*(S)$ by the formulas

$$\varphi(T_a) = T_{\tau(a)}, \quad \varphi(T_a^*) = T_{\tau(a)}^*, \tag{12}$$

can be extended to the whole C^*-algebra $C_r^*(S)$.

Let $X \subset L \setminus \tau(S)$ be a generating set for the extension (L, τ, σ). Recall that X consists of all elements x_γ such that $X \cap L_\gamma = \{x_\gamma\}$ for every $\gamma \in \Gamma$, $\gamma \neq 0$.

Consider $l^2(L)$ and a subspace H_γ, $\gamma \in \Gamma$, of (11). By the definition of a normal extension with a generating set, each $y \in L \setminus \tau(S)$ is uniquely representable in the form $y = \tau(a) + x_\gamma$, where $a \in S, x_\gamma \in X$. Therefore, an orthonormal basis for H_γ is given by the set of functions $\{e_{\tau(a)+x_\gamma} \mid a \in S, x_\gamma \in X\}$ for $\gamma \neq 0$ and $\{e_{\tau(a)} \mid a \in S\}$ for $\gamma = 0$.

Given $\gamma \in \Gamma$, construct the unitary operator

$$U_\gamma : l^2(S) \longrightarrow H_\gamma$$

such that $U_0 e_a = e_{\tau(a)}$ if $\gamma = 0$ and $U_\gamma e_a = e_{\tau(a)+x_\gamma}$ if $\gamma \neq 0$.

Using Lemma 3.1, represent each $T_{\tau(a)}$ as the direct sum

$$T_{\tau(a)} = \bigoplus_{\gamma \in \Gamma} T_{\tau(a)}^\gamma,$$

where $T_{\tau(a)}^\gamma = T_{\tau(a)}|_{H_\gamma}$ stands for the restriction of $T_{\tau(a)}$ to H_γ.

Likewise, $V_\tau \in \mathrm{Mon}_\tau$ and $A_\tau \in P(\tau(S))$ are representable as the direct sums

$$V_\tau = \bigoplus_{\gamma \in \Gamma} V_\tau^\gamma, \quad A_\tau = \bigoplus_{\gamma \in \Gamma} A_\tau^\gamma, \tag{13}$$

where $V_\tau^\gamma = V_\tau|_{H_\gamma}$ and $A_\tau^\gamma = A_\tau|_{H_\gamma}$ are the corresponding restrictions to the subspace H_γ.

Fix $\gamma \in \Gamma$ and show that the diagram

$$\begin{tikzcd} l^2(S) \arrow[r, "T_a"] \arrow[d, "U_\gamma"'] & l^2(S) \arrow[d, "U_\gamma"] \\ H_\gamma \arrow[r, "T_{\tau(a)}^\gamma"'] & H_\gamma \end{tikzcd}$$

commutes; i.e.,

$$T^{\gamma}_{\tau(a)}U_{\gamma} = U_{\gamma}T_a. \tag{14}$$

Indeed, for all $c \in S$ and $\gamma \neq 0$,

$$T^{\gamma}_{\tau(a)}U_{\gamma}e_c = T^{\gamma}_{\tau(a)}e_{\tau(c)+x_{\gamma}} = e_{\tau(a)+\tau(c)+x_{\gamma}} = U_{\gamma}e_{a+c} = U_{\gamma}T_a e_c.$$

If $\gamma = 0$ then

$$T^0_{\tau(a)}U_0 e_c = T^0_{\tau(a)}e_{\tau(c)} = e_{\tau(a)+\tau(c)} = U_0 e_{a+c} = U_0 T_a e_c.$$

Apply involution to (14). Then, for every $\gamma \in \Gamma$, we obtain

$$(T^*_{\tau(a)})^{\gamma}U_{\gamma} = U_{\gamma}T^*_a. \tag{15}$$

It is not hard to see that operators of the form (7) and (9) satisfy the analogous equality

$$V^{\gamma}_{\tau}U_{\gamma} = U_{\gamma}V. \tag{16}$$

To this end, it suffices to apply (14) and (15), which yields

$$V^{\gamma}_{\tau}U\gamma = \big(T^{i_n}_{\tau(a_n)}\big)^{\gamma}\big(T^{i_{n-1}}_{\tau(a_{n-1})}\big)^{\gamma}\dots\big(T^{i_1}_{\tau(a_1)}\big)^{\gamma}U_{\gamma} = U_{\gamma}T^{i_n}_{a_n}T^{i_{n-1}}_{a_{n-1}}\dots T^{i_1}_{a_1} = U_{\gamma}V.$$

Finally, using (16), we obtain a relation for operators of the form (8) and (10):

$$A^{\gamma}_{\tau}U_{\gamma} = \sum_{i=1}^{m}\alpha_i(V_i)^{\gamma}_{\tau}U_{\gamma} = \sum_{i=1}^{m}\alpha_i U_{\gamma}V_i = U_{\gamma}A. \tag{17}$$

Extend φ acting at the generators as in (12) to operators V of the form (7) as follows:

$$\varphi(V) = V_{\tau}.$$

Prove the correctness of this extension; i.e., assuming that $V_1 = V_2$ on $l^2(S)$ show that $\varphi(V_1) = \varphi(V_2)$ on $l^2(L)$.

Assume that $V_1e_c = V_2e_c$ for every $c \in S$. Then $(V_1)_{\tau}e_y = (V_2)_{\tau}e_y$ for every $y \in L$. Recall that, by the definition of a normal extension with a generating set, each $y \in L$ is uniquely representable as $y = \tau(c) + x_{\gamma}$, where $c \in S$ and $x_{\gamma} \in X$. By (16) for every $y \in L$ we obtain

$$(V_1)_{\tau}e_y = (V_1)^{\gamma}_{\tau}e_{\tau(c)+x_{\gamma}} = (V_1)^{\gamma}_{\tau}U_{\gamma}e_c = U_{\gamma}V_1e_c = U_{\gamma}V_2e_c = (V_2)^{\gamma}_{\tau}U_{\gamma}e_c = (V_2)^{\gamma}_{\tau}e_{\tau(c)+x_{\gamma}} = (V_2)_{\tau}e_y.$$

Thus, $(V_1)_{\tau} = (V_2)_{\tau}$, i.e., $\varphi(V_1) = \varphi(V_2)$.

Now, extend φ to the finite linear combinations A of the form (8) as follows: $\varphi(A) = A_{\tau}$. Using (17), we similarly prove the correctness of this extension.

The so-constructed mapping φ is a unital $*$-homomorphism from $P(S)$ into the C^*-algebra $C^*_r(L)$.

Equalities (13) and (17) imply that each operator A_{τ} is representable as the direct sum

$$A_{\tau} = \bigoplus_{\gamma\in\Gamma}U_{\gamma}AU^*_{\gamma}.$$

Thus, $\|A_{\tau}\| = \|A\|$. Consequently, φ is an isometric $*$-homomorphism on the algebra $P(S)$ dense in $C^*_r(S)$. This means that φ can be extended uniquely to an isometric $*$-homomorphism on the whole C^*-algebra $C^*_r(S)$. $\square$

In conclusion, we give the example showing that if a normal extension (L, τ, σ) of a semigroup S by some group Γ has no generating set; then the correspondence $T_a \longmapsto T_{\tau(a)}$ does not extend in general to an injective $*$-homomorphism from $C^*_r(S)$ into $C^*_r(L)$. In other words, an embedding of $C^*_r(S)$ into the C^*-algebra $C^*_r(L)$ generated by an isometric representation of S may fail to exist.

EXAMPLE 3.1. As the semigroup L, consider the set

$$L = Z^{2,3} \sqcup \{m + 3/2 \mid m \in \mathbb{Z}^+\}$$

with the usual addition. Consider the short exact sequence

$$0 \longrightarrow Z^{2,3} \xrightarrow{\tau} L \xrightarrow{\sigma} \mathbb{Z}_2 \longrightarrow 0,$$

in which the semigroup homomorphisms are defined as follows: $\tau(n) = n$, $\sigma(n) = [0]_2$, and $\sigma\left(m + \frac{3}{2}\right) = [1]_2$ for all $n \in Z^{2,3}$ and $m \in \mathbb{Z}^+$. Obviously, $\sigma^{-1}([0]_2) = Z^{2,3}$. Thus, the triple (L, τ, σ) is a normal extension of $Z^{2,3}$ by $\mathbb{Z}_2$.

Show that this extension has no generating set. Since $\mathbb{Z}_2$ is a finite group of order 2, any generating set, if it exists, must consist of a single element. Obviously, this must be the least element of the set $\{m + 3/2 \mid m \in \mathbb{Z}^+\}$, i.e., $\frac{3}{2}$. But then $\frac{5}{2}$ is not representable as $n + \frac{3}{2}$, where $n \in Z^{2,3}$.

Let us show that there is no $*$-homomorphism $\varphi : C_r^*(Z^{2,3}) \longrightarrow C_r^*(L)$ such that the diagram

$$\begin{tikzcd} Z^{2,3} \arrow[r, "\tau"] \arrow[d, "\pi_{Z^{2,3}}"'] & L \arrow[d, "\pi_L"] \\ C_r^*(Z^{2,3}) \arrow[r, "\varphi"'] & C_r^*(L) \end{tikzcd}$$

commutes, where $\pi_{Z^{2,3}}(n) = T_n$, $\pi_L(k) = T_k$, $n \in Z^{2,3}$, $k \in L$. Suppose that such a homomorphism φ exists. Then we must have $\varphi(T_n) = T_{\tau(n)}$ for every $n \in Z^{2,3}$.

Consider the following operators in $C_r^*(Z^{2,3})$:

$$P_0 = I - T_3^* T_2 T_2^* T_3, \quad P_3 = T_3 P_0 T_3^*, \quad P_{0,3} = I - T_2 T_2^*, \quad Q = P_{0,3} - P_3 - P_0.$$

It is easy to notice that P_0 is the projection to the one-dimensional subspace with basis $\{e_0\}$ in $l^2(Z^{2,3})$. Also, P_3 is a projection to the subspace with basis $\{e_3\}$ and $P_{0,3}$ is the projection to the subspace with basis $\{e_0, e_3\}$. Therefore, $Q = 0$.

Consider the operator $\varphi(Q) = \varphi(P_{0,3}) - \varphi(P_3) - \varphi(P_0)$ which is an element of $C_r^*(L)$. Recall that $l^2(L)$ is representable as the direct sum

$$l^2(L) = H_0 \oplus H_1,$$

where H_0 has the basis $\{e_n\}_{n \in Z^{2,3}}$ and H_1 has the basis $\{e_{m+\frac{3}{2}}\}_{m \in Z^+}$. Then $\varphi(Q)$ is representable as the direct sum

$$\varphi(Q) = \varphi(Q)|_{H_0} \oplus \varphi(Q)|_{H_1}.$$

Likewise, the operators $\varphi(P_0)$, $\varphi(P_3)$, and $\varphi(P_{0,3})$ are representable as direct sums. It is not hard to see that

$$\varphi(P_0)|_{H_1} = 0, \quad \varphi(P_3)|_{H_1} = 0$$

and $\varphi(P_{0,3})|_{H_1}$ is the projection to the subspace in H_1 with basis $\{e_{\frac{3}{2}}, e_{\frac{5}{2}}\}$. Therefore, $\varphi(Q)|_{H_1} \neq 0$ though $\varphi(Q)|_{H_0} = 0$. Thus, $\varphi(Q) \neq 0$; a contradiction.

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