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Inductive Systems of C^*-Algebras over Posets: A Survey

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Abstract—We survey the research on the inductive systems of C^*-algebras over arbitrary partially ordered sets. The motivation for our work comes from the theory of reduced semigroup C^*-algebras and local quantum field theory. We study the inductive limits for the inductive systems of Toeplitz algebras over directed sets. The connecting $*$-homomorphisms of such systems are defined by sets of natural numbers satisfying some coherent property. These inductive limits coincide up to isomorphisms with the reduced semigroup C^*-algebras for the semigroups of non-negative rational numbers. By Zorn's lemma, every partially ordered set K is the union of the family of its maximal directed subsets K_i indexed by elements of a set I. For a given inductive system of C^*-algebras over K one can construct the inductive subsystems over K_i and the inductive limits for these subsystems. We consider a topology on the set I. It is shown that characteristics of this topology are closely related to properties of the limits for the inductive subsystems.

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1. INTRODUCTION

The paper is a survey of the results on the inductive systems of C^*-algebras presented in [1–5]. These inductive systems are considered over arbitrary partially ordered sets.

The motivation for our work comes from several sources. On the one hand, these sources comprise the papers [6–15] on the reduced semigroup C^*-algebras which are also called the Toeplitz algebras and their inductive systems. Coburn [6, 7] and Douglas [8] studied the Toeplitz algebras for the subsemigroups of the additive group of real numbers. Murphy [9, 10] considered the general case of ordered groups. In particular, he studied the inductive systems of the Toeplitz algebras and proved that the correspondence between ordered groups and Toeplitz algebras is a continuous functor. These authors discovered that the isometric representations of semigroups possess the universal property. For the case of the semigroup of all non-negative integers this property is also known as Coburn's theorem (see, for example, [16, theorem 3.5.18]). Using this theorem, the inductive sequences of the Toeplitz algebras associated with arbitrary sequences of prime numbers are defined and studied in [1]. The results of [1] are connected with the properties of mappings between the topological groups which are considered in [17–23]. In [15] the authors constructed the inductive system making use of the reduced semigroup C^*-algebra which is generated by the representation of the path semigroup for a partially ordered set. On the other hand, the motivation comes from algebraic quantum field theory [24]. We recall that the general framework of this theory is given by a covariant functor acting from a category whose objects are

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topological spaces with additional structures and its morphisms are structure preserving embeddings. That functor takes its values in a category describing the algebraic structure of observables. The basic tool of the algebraic approach to quantum fields over a spacetime is a net of C^*-algebras over a partially ordered set defined as a suitable set of regions of the spacetime ordered under the inclusion. For the case of a curved spacetime, nets containing C^*-algebras of quantum observables are studied in [25–27]. The paper [28] is devoted to nets of C^*-algebras associated with nets over partially ordered sets consisting of Hilbert spaces.

The present paper deals with covariant functors acting from categories associated to arbitrary partially ordered sets into the category of unital C^*-algebras and their unital $*$-homomorphisms. As is well-known, those functors are also called inductive systems over posets.

We begin by considering the inductive systems of Toeplitz algebras over directed sets. The connecting $*$-homomorphisms of such systems are defined by sets of natural numbers satisfying the factorization equalities. The inductive limit of such an inductive system coincides up to an isomorphism with a reduced semigroup C^*-algebra for a semigroup of non-negative rational numbers.

By Zorn's lemma, every partially ordered set K can be represented as the union of the family $\{K_i\}$ of its maximal directed subsets indexed by elements of a set I. We consider a topology on the set I generated by a neighbourhood system. For every index $i \in I$ the original inductive system over K yields naturally the inductive system of C^*-algebras over K_i and its inductive limit. Using these inductive limits, we construct different types of C^*-algebras. In particular, for neighbourhoods of the topology on the set of indices we deal with the C^*-algebras which are the direct products of limits for inductive systems over the sets K_i. We survey properties of the above-mentioned topology and the C^*-algebras. It is shown that there exists a connection between the topological and algebraic structures.

The paper is organized as follows. It consists of Introduction and five more sections. In Section 2, after giving the necessary preliminaries on the inductive systems over arbitrary partially ordered sets and their limits in the category of unital C^*-algebras, we recall some facts about the reduced semigroup C^*-algebras for semigroups of rational numbers. Section 3 is devoted to the inductive systems of Toeplitz algebras over directed sets. The connecting $*$-homomorphisms of those systems are defined by arbitrary sets of natural numbers satisfying the factorization equalities. Section 4 is concerned with the topology on the index set I. The set I is endowed with the topology by means of a neighbourhood system. We list some properties of this topology and give examples of such topological spaces. Section 5 deals with inductive systems of arbitrary unital C^*-algebras over partially ordered sets. For a given inductive system over a poset K we consider the inductive subsystems over the maximal directed subsets K_i and their limits $\mathfrak{A}^{K_i}$. Using the C^*-algebras $\mathfrak{A}^{K_i}$ and the neighbourhoods of the topology on I, we construct a new inductive system of C^*-algebras over K and its inductive subsystems over K_i with the limits denoted by $\mathfrak{B}^{K_i}$. Making use of properties of the topology on the index set I, we discuss the relationship between the C^*-algebras $\mathfrak{A}^{K_i}$ and $\mathfrak{B}^{K_i}$. In Section 6 we consider the inductive systems of Toeplitz algebras over partially ordered sets. It is shown that the reduced semigroup C^*-algebras for the semigroups of non-negative rational numbers are closely connected with such inductive systems.

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2. PRELIMINARIES

Throughout the paper $(K, \leq)$ stands for a partially ordered set. In general, we do not assume that it is a directed set. The category associated to this set is denoted by the same letter K. We recall that the objects of this category are the elements of the set K, and, for any pair $a, b \in K$, the set of morphisms from a to b consists of the single element (a, b) provided that $a \leq b$, and is the void set otherwise.

Together with a partially ordered set $(K, \leq)$ we shall consider a covariant functor $\mathcal{F}$ from the category K into the category of unital C^*-algebras and their unital $*$-homomorphisms. Such a functor is called *an inductive system* in the category of C^*-algebras over the set $(K, \leq)$. It may be given by a system $(K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$ satisfying the standard conditions in the definition of a covariant functor. In the sequel, we shall write

$$\mathcal{F} = (K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\}). \tag{1}$$

Here, $\{\mathfrak{A}_a | a \in K\}$ is a family of unital C^*-algebras. For a unital C^*-algebra $\mathfrak{A}$ its unit will be denoted by $\mathbb{I}_{\mathfrak{A}}$. We also suppose that all morphisms $\sigma_{ba} : \mathfrak{A}_a \longrightarrow \mathfrak{A}_b$, where $a \leq b$, are embeddings of C^*-algebras, i. e., unital injective $*$-homomorphisms. Recall that the diagram

$$\begin{tikzcd} \mathfrak{A}_a \arrow[rr, "\sigma_{ca}"] \arrow[dr, "\sigma_{ba}"'] & & \mathfrak{A}_c \\ & \mathfrak{A}_b \arrow[ur, "\sigma_{cb}"'] & \end{tikzcd}$$

commutes for all elements $a, b, c \in K$ satisfying the conditions $a \leq b$ and $b \leq c$, that is, the following equality holds:

$$\sigma_{ca} = \sigma_{cb} \circ \sigma_{ba}. \tag{2}$$

It is worth recalling that for each element $a \in K$ the morphism $\sigma_{aa} : \mathfrak{A}_a \longrightarrow \mathfrak{A}_a$ is the identity mapping.

Further, let us consider the family of all upward directed subsets of the set $(K, \leq)$. Making use of Zorn's lemma, one can easily prove the following statement.

Lemma 1. *Let $(K, \leq)$ be a partially ordered set. Then the following equality holds:*

$$K = \bigcup_{i \in I} K_i, \tag{3}$$

where $\{K_i | i \in I\}$ is the family of all maximal upward directed subsets of $(K, \leq)$.

Moreover, for every $i \in I$ and $a \in K_i$ the set $\{b \in K | b \leq a\}$ is a subset of K_i.

For each index $i \in I$ we may consider the inductive system $\mathcal{F}_i = (K_i, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$ over the maximal upward directed set K_i.

The simplest example of the inductive system $\mathcal{F}_i$ is that in which $\{\mathfrak{A}_a | a \in K_i\}$ is *a net of C^*-subalgebras* of a given C^*-algebra $\mathfrak{A}$. By this, one means that each $\mathfrak{A}_a$ is a C^*-subalgebra containing the unit $\mathbb{I}_{\mathfrak{A}}$, $\mathfrak{A}_a \subset \mathfrak{A}_b$ and $\sigma_{ba} : \mathfrak{A}_a \longrightarrow \mathfrak{A}_b$ is the inclusion mapping whenever $a, b \in K_i$ and $a \leq b$. Given such a net $\mathcal{F}_i$, the norm closure of the union of all $\mathfrak{A}_a$ is itself a C^*-subalgebra of $\mathfrak{A}$ that is called the inductive limit of the net $\mathcal{F}_i$.

We recall the definition and some facts concerning the inductive limits for inductive systems of C^*-algebras (see, for example, [29, Section 11.4]).

The inductive limit of the inductive system $\mathcal{F}_i = (K_i, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$ is a pair $(\mathfrak{A}^{K_i}, \{\sigma_a^{K_i}\})$, where $\mathfrak{A}^{K_i}$ is a C^*-algebra and $\{\sigma_a^{K_i} : \mathfrak{A}_a \to \mathfrak{A}^{K_i} | a \in K_i\}$ is a family of unital injective $*$-homomorphisms such that the following two properties are fulfilled [29, Proposition 11.4.1]:

1) For every pair elements $a, b \in K_i$ satisfying the condition $a \leq b$ the diagram

$$\begin{tikzcd} \mathfrak{A}_a \arrow[rr, "\sigma_{ba}"] \arrow[dr, "\sigma_a^{K_i}"'] & & \mathfrak{A}_b \arrow[dl, "\sigma_b^{K_i}"] \\ & \mathfrak{A}^{K_i} & \end{tikzcd}$$

is commutative, that is, the equality for mappings

$$\sigma_a^{K_i} = \sigma_b^{K_i} \circ \sigma_{ba} \tag{4}$$

holds. Moreover, we have the following equality:

$$\mathfrak{A}^{K_i} = \overline{\bigcup_{a \in K_i} \sigma_a^{K_i}(\mathfrak{A}_a)}, \tag{5}$$

where the bar means the closure of the set with respect to the norm topology in the C^*-algebra $\mathfrak{A}^{K_i}$.

2) *The universal property.* If $\mathfrak{B}$ is a C^*-algebra, $\psi_a : \mathfrak{A}_a \longrightarrow \mathfrak{B}$ is an injective $*$-homomorphism for each $a \in K_i$, and the conditions analogous to those in (4) and (5) are satisfied, i.e., the diagram

$$\begin{tikzcd} \mathfrak{A}_a \arrow[rr, "\sigma_{ba}"] \arrow[dr, "\psi_a"'] & & \mathfrak{A}_b \arrow[dl, "\psi_b"] \\ & \mathfrak{B} & \end{tikzcd}$$

commutes for each pair $a, b \in K_i$ satisfying the condition $a \leq b$ and the equality

$$\mathfrak{B} = \overline{\bigcup_{a \in K_i} \psi_a(\mathfrak{A}_a)}$$

holds, then there exists a unique $*$-isomorphism θ from $\mathfrak{A}^{K_i}$ onto $\mathfrak{B}$ such that the diagram

$$\begin{tikzcd} & \mathfrak{A}_a \arrow[dl, "\sigma_a^{K_i}"'] \arrow[dr, "\psi_a"] & \\ \mathfrak{A}^{K_i} \arrow[rr, "\theta"] & & \mathfrak{B} \end{tikzcd}$$

is commutative for every $a \in K_i$, that is, the equality $\psi_a = \theta \circ \sigma_a^{K_i}$ holds. The inductive limit $(\mathfrak{A}^{K_i}, \{\sigma_a^{K_i}\})$ of the inductive system $\mathcal{F}_i$ is denoted as follows:

$$(\mathfrak{A}^{K_i}, \{\sigma_a^{K_i}\}) := \varinjlim \mathcal{F}_i.$$

The limit C^*-algebra $\mathfrak{A}^{K_i}$ itself is often called the inductive limit of the inductive system $\mathcal{F}_i$. For this algebra we also use notation

$$\mathfrak{A}^{K_i} := \varinjlim \mathcal{F}_i. \tag{6}$$

As is well-known, an inductive limit for an inductive system can always be constructed in the category of unital C^*-algebras and their $*$-homomorphisms.

If we are given the covariant functor $\mathcal{F}$, then we can construct the direct product for the inductive limits of the functors $\mathcal{F}_i$ which is denoted by the symbol

$$\mathfrak{M}_{\mathcal{F}} := \prod_{i \in I} \mathfrak{A}^{K_i} = \left\{ (a_i) \big| ||(a_i)|| = \sup_i ||a_i|| < \infty \right\}. \tag{7}$$

It is easy to check that the direct product $\mathfrak{M}_{\mathcal{F}}$ is a unital C^*-algebra with respect to the pointwise operations and the supremum norm. We say that the C^*-algebra contains information about all inductive limits of subsystems over maximal directed subsets for inductive system (1).

Further, we recall the definition of the reduced semigroup C^*-algebras for semigroups in the additive group of all rational numbers $\mathbb{Q}$.

Assume that Γ is an arbitrary subgroup in $\mathbb{Q}$. Let $\Gamma^+ := \Gamma \cap [0, +\infty)$ be the positive cone in the ordered group Γ. As usual, the symbol $l^2(\Gamma^+)$ stands for the Hilbert space of all square summable complex-valued functions on the additive subgroup Γ^+. The canonical orthonormal basis in the Hilbert space $l^2(\Gamma^+)$ is denoted by $\{e_g | g \in \Gamma^+\}$. That is, for all elements $g, h \in \Gamma^+$, we set $e_g(h) = \delta_{g,h}$, where $\delta_{g,h} = 1$ if $g = h$, and $\delta_{g,h} = 0$ if $g \neq h$.

Let us consider the C^*-algebra of all bounded linear operators $B(l^2(\Gamma^+))$ in the Hilbert space $l^2(\Gamma^+)$. For every element $g \in \Gamma^+$, we define the isometry $V_g \in B(l^2(\Gamma^+))$ by

$$V_g e_h = e_{g+h},$$

where $h \in \Gamma^+$.

Definition 1. *The C^*-subalgebra in the algebra $B(l^2(\Gamma^+))$ generated by the set $\{V_g | g \in \Gamma^+\}$ is called the reduced semigroup C^*-algebra of the semigroup Γ^+, or the Toeplitz algebra generated by Γ^+. It is denoted by $C_r^*(\Gamma^+)$.*

In the case when Γ is the group of all integers $\mathbb{Z}$, we also denote the semigroup C^*-algebra $C_r^*(\mathbb{Z}^+)$ by $\mathcal{T}$ and use the symbols T and T^n instead of V_1 and V_n, respectively, where $n \in \mathbb{Z}^+$.

In the similar way a semigroup C^*-algebra can be defined for an arbitrary cancellative semigroup. As is noted in [30, Section 2], a semigroup C^*-algebra is a very natural object. It is generated by the left regular representation of a given semigroup.

The following lemma is an immediate consequence of Coburn's theorem [16, Theorem 3.5.18]. It is worth noting that a straightforward proof of this lemma is also given in [31].

Lemma 2. *For every number $n \in \mathbb{N}$, there exists a unique unital $*$-homomorphism of C^*-algebras $\varphi : \mathcal{T} \longrightarrow \mathcal{T}$ such that $\varphi(T) = T^n$. Moreover, φ is isometric.*

In the sequel, we abbreviate those self-homomorphisms of the Toeplitz algebra as follows:

$$\varphi : \mathcal{T} \longrightarrow \mathcal{T} : T \longmapsto T^n.$$

Let $M = (m_1, m_2, m_3, ...)$ be an arbitrary sequence of positive integers. In what follows, we shall consider the reduced semigroup C^*-algebra $C^*_r(Q^+_M)$ for the semigroup of non-negative rational numbers

$$Q^+_M = \left\{ \frac{m}{m_1 \cdot m_2 \cdot ... \cdot m_n} \Big| m \in \mathbb{Z}^+, n \in \mathbb{N} \right\}.$$

For necessary results in the theory of C^*-algebras we refer the reader, for example, to the books [16] and [32, Ch. 4, § 7].

3. INDUCTIVE SYSTEMS OF TOEPLITZ ALGEBRAS OVER DIRECTED SETS

The notion of the inductive sequence of the Toeplitz algebras defined by an arbitrary sequence of prime numbers is introduced in the paper [1]. In [4] this notion is generalized for the inductive systems of the Toeplitz algebras over directed sets. The connecting $*$-homomorphisms of those systems are determined by sets of natural numbers satisfying the factorization equalities. Here, we discuss the results on the inductive limits of such systems.

Throughout the section we assume that $(K, \leq)$ is a directed set. Let us consider a set of natural numbers

$$N = \{n_{ba} \in \mathbb{N} | a, b \in K, a \leq b\} \tag{8}$$

such that the factorization equalities

$$n_{ca} = n_{cb} \cdot n_{ba} \tag{9}$$

hold for all elements $a, b, c \in K$ satisfying the conditions $a \leq b$ and $b \leq c$. It follows immediately from (9) that the equality $n_{aa} = 1$ is valid for every $a \in K$.

Further, using Lemma 2, for every number $n_{ba} \in N$ we define the isometric $*$-homomorphism by the formula

$$\sigma_{ba} : \mathcal{T} \longrightarrow \mathcal{T} : T \longmapsto T^{n_{ba}}. \tag{10}$$

It is clear that equalities (2) are valid for all elements $a, b, c \in K$ whenever the conditions $a \leq b$ and $b \leq c$ hold, and for each $a \in K$ the $*$-homomorphism $\sigma_{aa} : \mathfrak{A}_a \longrightarrow \mathfrak{A}_a$ is the identity mapping. Thus we can give the following definition of the inductive system of the Toeplitz algebras over a directed set defined by a set of natural numbers satisfying the factorization equalities.

Definition 2. *Let $(K, \leq)$ be a directed set and N be a set of natural numbers (8) satisfying (9). An inductive system of C^*-algebras*

$$\mathcal{F} = (K, \{\mathcal{T}_a\}, \{\sigma_{ba}\}), \tag{11}$$

where $\mathcal{T}_a = \mathcal{T}$ for all $a \in K$ and the connecting $*$-homomorphisms σ_{ba} are given by (10), is called the inductive system of Toeplitz algebras over K defined by N.

The inductive systems over subsets in $(K, \leq)$ that are constructed from the elements of inductive system (11) will be called the inductive subsystems of (11).

We have the following result that generalizes Proposition 1 in [1].

Theorem 1. [4, Theorem 1] *Let $\mathcal{F}$ be an inductive system of Toeplitz algebras over a directed set K defined by a set of natural numbers N satisfying the factorization equalities. Then there exists a semigroup of non-negative rational numbers $\mathbb{Q}^+_M$ such that*

$$C^*_r(\mathbb{Q}^+_M) \simeq \varinjlim \mathcal{F}. \tag{12}$$

To prove isomorphism (12) in the category of C^*-algebras we proceed as follows.

For an element $a \in K$ we define the subset K^a in the directed set K by

$$K^a := \{b \in K | a \leq b\}. \tag{13}$$

Obviously, the subset K^a is cofinal in K. Furthermore, together with set (13) we consider the inductive system

$$(K^a, \{\mathcal{T}_b\}, \{\sigma_{cb}\}) \tag{14}$$

of Toeplitz algebras over K^a defined by the set

$$\{n_{cb} \in N | b, c \in K^a\}\,. \tag{15}$$

Using the universal property for inductive subsystem (14) defined by (15) one has immediately the following property.

Lemma 3. *For every element $a \in K$ there exists an isomorphism between the inductive limits*

$$\varinjlim(K^a, \{\mathcal{T}_b\}, \{\sigma_{cb}\}) \simeq \varinjlim(K, \{\mathcal{T}_b\}, \{\sigma_{cb}\}) \tag{16}$$

in the category of unital C^*-algebras and their unital $*$-homomorphisms.

The next step for proving isomorphism (12) is the construction of a totally ordered countable subset Λ^a in the set K^a satisfying some additional properties. It is worth noting that the set Λ^a is finite or infinite. For details of this construction we refer the reader to [4, Section 3]. We denote the elements of Λ^a by c_s, where $s \in \mathbb{N}$, and consider the inductive sequence of Toeplitz algebras

$$(\Lambda^a, \{\mathcal{T}_{c_s}\}, \{\sigma_{c_t c_s}\}) \tag{17}$$

over the set Λ^a defined by the subset $\{n_{cb} \in N | b, c \in \Lambda^a\}$ in set (8). We note that sequence (17) is an inductive subsystem of both inductive systems (11) and (14).

The following statement is an analog of Proposition 1 in [1].

Lemma 4. *There exists a semigroup $\mathbb{Q}^+_M$ of non-negative rational numbers such that the following isomorphism holds in the category of unital C^*-algebras and their unital $*$-homomorphisms:*

$$C^*_r(\mathbb{Q}^+_M) \simeq \varinjlim\,(\Lambda^a, \{\mathcal{T}_{c_s}\}, \{\sigma_{c_t c_s}\})\,. \tag{18}$$

Making use of the universal property for inductive sequence (17) one obtains the following isomorphism of inductive limits.

Lemma 5. *There exists an isomorphism between the inductive limits*

$$\varinjlim\,(\Lambda^a, \{\mathcal{T}_{c_s}\}, \{\sigma_{c_t c_s}\}) \simeq \varinjlim\,(K^a, \{\mathcal{T}_b\}, \{\sigma_{cb}\})\,. \tag{19}$$

in the category of unital C^*-algebras and their unital $*$-homomorphisms.

Finally, combining isomorphisms (16), (18) and (19), we get isomorphism (12), as claimed.

4. TOPOLOGY ON THE INDEX SET

In what follows, $(K, \leq)$ is an arbitrary partially ordered set that is not necessarily directed. By Lemma 1, we have representation (3) of the set K as the union of all maximal upward directed subsets K_i, $i \in I$.

In this section we consider the topology on the index set I which was introduced in [2, 3]. The topological space I is closely related to properties of inductive limits for subsystems of an inductive system over the set $(K, \leq)$. We describe the topology and give several examples of index sets supplied with that topology.

First, let us recall briefly the definition of this topology and its properties. For every element $a \in K$ we define the subset U_a in the index set I by the formula

$$U_a := \{i \in I | a \in K_i\}\,.$$

The sets U_a, $a \in K$, possess the following properties [3, Lemma 1, Propositions 2 and 4]:

—If $a, b \in K$ such that $a \leq b$, then $U_b \subset U_a$;

—The family $\{U_a | a \in K\}$ constitutes a base for a topology on the index set I;

—For $a \in K$, the set K^a defined in (13) is upward directed if and only if the neighbourhood U_a is an one-point set.

In the sequel, we denote by τ the topology on the index set I generated by the base $\{U_a | a \in K\}$. We have the following statement on the topology τ.

Proposition 1 [3, Proposition 3]. *The topological space (I, τ) is a T_1-space.*

This proposition immediately implies the corollaries.

Corollary 1. *For every index $i \in I$ the equality $\bigcap\limits_{a \in K_i} U_a = \{i\}$ holds.*

Corollary 2. *For every index $i \in I$ the one-point set $\{i\}$ is closed.*

To demonstrate various properties of the topology τ we give examples of various topological spaces (I, τ) which were constructed in [2, 3, 5].

The first example shows that (I, τ) may not be a Hausdorff space.

Example 1. On the plane we consider the set of points with integer coordinates

$$K := \{(x, y) | x \in \{-1; 0; 1\}, y \in \mathbb{Z}\}.$$

A partial order $\leq$ on the set K is defined as follows:

$$(x_1, y_1) \leq (x_2, y_2) := \begin{cases} x_1, x_2 \in \{-1; 1\}, \quad x_1 = x_2, \quad y_1 \leq y_2; \\ x_1 \in \{-1; 1\}, \quad x_2 = 0, \quad y_1 < y_2. \end{cases}$$

It is straightforward to check that the pair $(K, \leq)$ is a partially ordered set, which is not upward directed.

One has the representation of K as the union of maximal upward directed sets K_i indexed by the set of all integers $\mathbb{Z}$ together with two symbols $-\infty$ and $+\infty$, that is, $I = \mathbb{Z} \cup \{-\infty; +\infty\}$:

$$K = \bigcup_{i=-\infty}^{+\infty} K_i, \quad \text{where} \quad K_{-\infty} := \{(-1, y) | y \in \mathbb{Z}\}; \quad K_{+\infty} := \{(1, y) | y \in \mathbb{Z}\}$$

$$\text{and} \quad K_i := \{(0, i)\} \bigcup \{(x, y) | x \in \{-1; 1\}, y < i, y \in \mathbb{Z}\} \quad \text{for each} \quad i \in \mathbb{Z}.$$

A base $\{U_{(x,y)} | x \in \{-1; 0; 1\}, y \in \mathbb{Z}\}$ for the topology τ on the index set I consists of the sets of three types, namely,

$$U_{(-1,y)} := \{-\infty\} \cup \{i \in \mathbb{Z} | i > y\}; U_{(1,y)} := \{+\infty\} \cup \{i \in \mathbb{Z} | i > y\}; U_{(0,y)} := \{y\}.$$

Since any two neighbourhoods of indices $-\infty$ and $+\infty$ have a non-empty intersection the topological space (I, τ) is not a Hausdorff space.

The second example yields a locally compact Hausdorff topological space (I, τ).

Example 2. As the set K we consider the lower half-plane without the axis $y = 0$, that is,

$$K = \{(x, y) | x, y \in \mathbb{R}, y < 0\}.$$

We define a partial order $\leq$ on K as follows. Let us fix a positive number $a \in \mathbb{R}$. Then we put

$$(x_1, y_1) \leq (x_2, y_2) := \begin{cases} x_1 = x_2 \quad \text{and} \quad y_1 = y_2; \\ y_2 - y_1 > a|x_2 - x_1|. \end{cases}$$

It is easily verified that the pair $(K, \leq)$ is a partially ordered set. Moreover, it is worth noting that this set is not upward directed.

We have the representation of K as the union of maximal upward directed sets K_i indexed by the set of all real numbers, that is, $I = \mathbb{R}$:

$$K = \bigcup_{i \in \mathbb{R}} K_i, \quad \text{where} \quad K_i := \{(x, y) \in K | -y > a|i - x|\}.$$

Taking a point $(x_0, y_0) \in K$, one can easy see that

$$U_{(x_0,y_0)} = \Big\{ i \in \mathbb{R} | x_0 + \frac{y_0}{a} < i < x_0 - \frac{y_0}{a} \Big\}.$$

Thus, in this example the topology τ coincides with the natural topology on the set of real numbers $\mathbb{R}$ which is non-discrete locally compact.

A slight modification of Example 2 allows us to build an example of a partially ordered set $(K, \leq)$ such that $I = \mathbb{Z}$ and the topology τ is discrete. For this example we refer the reader to [3, Example 3].

The third example shows that the topological space (I, τ) may be compact.

Example 3. As the set K we consider the set of all closed arcs in the unit circle S^1:

$$K := \left\{ A \subset S^1 \middle| A = [e^{2\pi i x}, e^{2\pi i y}] \text{ or } A = S^1 \setminus (e^{2\pi i x}, e^{2\pi i y}), \text{ where } x, y \in [0,1) \text{ and } x < y \right\}.$$

A partial order on K is defined in the following way: for $A, B \in K$ we put $A \leq B \Leftrightarrow A \subset B$.

It is easily verified that the pair $(K, \leq)$ is a partially ordered set. Moreover, it is worth noting that this set is not directed. Indeed, take any $x_1, x_2, x_3, x_4 \in [0,1)$ such that $x_1 < x_2 < x_3 < x_4$. Then for $A = [e^{2\pi i x_1}, e^{2\pi i x_4}]$ and $B = S^1 \setminus (e^{2\pi i x_2}, e^{2\pi i x_3})$ there is no arc $C \in K$ such that $A \leq C$ and $B \leq C$.

One has the representation of K as the union of maximal upward directed sets K_z indexed by the points of the unit circle S^1, that is, $K = \bigcup\limits_{z \in S^1} K_z$, where $z = e^{2\pi i x}$, $x \in [0,1)$, and

$$K_z := \left\{ A \in K \middle| A \subset S^1 \setminus \{z\} \right\}.$$

A base $\{U_A | A \in K\}$ for the topology τ on the index set $I = S^1$ consists of the sets

$$U_A = \left\{ z \in S^1 \middle| A \in K_z \right\} = S^1 \setminus A.$$

Thus, the elements of the base for the topology τ are all open arcs of the unit circle S^1:

$$\left\{ B \subset S^1 \middle| B = (e^{2\pi i x}, e^{2\pi i y}) \text{ or } A = S^1 \setminus [e^{2\pi i x}, e^{2\pi i y}], \text{ where } x, y \in [0,1) \quad \text{and } x < y \right\}.$$

The topology τ coincides with the natural topology on the unit circle S^1 that is compact.

5. INDUCTIVE SYSTEMS OF ARBITRARY C^*-ALGEBRAS OVER POSETS

This section deals with inductive systems consisting of arbitrary unital C^*-algebras over partially ordered sets. We discuss the relation between the topological space (I, τ) introduced in the previous section and properties of C^*-algebras which are naturally associated with an inductive system over a partially ordered set in the category of unital C^*-algebras. In topological terms we give sufficient conditions for the existence of isomorphisms between C^*-algebras. Moreover, it is shown that the properties of the topological space (I, τ) are reflected in the structure of C^*-algebras.

We are given an inductive system (1), that is, $\mathcal{F} = (K, \{\mathfrak{A}_a\}, \{\sigma_{ba}\})$, where $\mathfrak{A}_a$ is an arbitrary unital C^*-algebra. For the partially ordered set K we consider union (3) of its maximal upward directed subsets K_i indexed by a set I. Then for each index $i \in I$ we can construct the inductive subsystem $\mathcal{F}_i$ of (1) over K_i and its inductive limit $\mathfrak{A}^{K_i}$ defined by (5).

Further, we take any $a \in K$ and consider the direct product of C^*-algebras

$$\mathfrak{B}_a := \prod_{i \in U_a} \mathfrak{A}^{K_i}. \tag{20}$$

We recall that for every pair $a, b \in K$ satisfying the condition $a \leq b$, we have the inclusion $U_b \subseteq U_a$ (see Section 4). Hence, the $*$-homomorphism $\tau_{ba} : \mathfrak{B}_a \longrightarrow \mathfrak{B}_b$ between C^*-algebras given by the rule

$$\tau_{ba}(f)(j) = f(j) \tag{21}$$

is well-defined. Here, we take $f \in \mathfrak{B}_a$ and $j \in U_b$.

It is straightforward to check that we have the equality $\tau_{ca} = \tau_{cb} \circ \tau_{ba}$ whenever $a, b, c \in K$ such that the both conditions $a \leq b$ and $b \leq c$ are fulfilled.

Thus we have constructed the inductive system of C^*-algebras and their $*$-homomorphisms

$$(K, \{\mathfrak{B}_a\}, \{\tau_{ba}\}). \tag{22}$$

Therefore, for each index $i \in I$ we can consider the inductive subsystem $(K_i, \{\mathfrak{B}_a\}, \{\tau_{ba}\})$ of system (22) and its inductive limit

$$(\mathfrak{B}^{K_i}, \{\tau_a^{K_i}\}) := \varinjlim (K_i, \{\mathfrak{B}_a\}, \{\tau_{ba}\}). \tag{23}$$

The following theorem shows the relation between two inductive limits (6) and (23).

Theorem 2. [3, Theorem 1] *For every index $i \in I$, the C^*-algebra $\mathfrak{A}^{K_i}$ is isomorphic to a C^*-subalgebra of $\mathfrak{B}^{K_i}$.*

Using the topology τ on the index set, we can guarantee in some cases the existence of isomorphism for inductive limits (6) and (23).

Theorem 3. [3, Theorem 2] *Let $i \in I$ be an isolated point in the topological space (I, τ). Then there exists an isomorphism*

$$\mathfrak{A}^{K_i} \simeq \mathfrak{B}^{K_i}$$

in the category of unital C^*-algebras and their unital $*$-homomorphisms.

For the direct product containing information about all inductive limits of subsystems over maximal directed subsets K_i for inductive system (22) we introduce the following notation:

$$\widehat{\mathfrak{M}}_{\mathcal{F}} := \prod_{i \in I} \mathfrak{B}^{K_i}. \tag{24}$$

As consequences of Theorems 2 and 3, one has respectively the following statements for C^*-algebraic direct products (7) and (24).

Corollary 3. *The C^*-algebra $\mathfrak{M}_{\mathcal{F}}$ is isomorphic to a C^*-subalgebra of $\widehat{\mathfrak{M}}_{\mathcal{F}}$.*

Corollary 4. *Let (I, τ) be a discrete topological space. Then there exists an isomorphism*

$$\mathfrak{M}_{\mathcal{F}} \simeq \widehat{\mathfrak{M}}_{\mathcal{F}}$$

in the category of unital C^*-algebras and their unital $*$-homomorphisms.

Further, it is shown that some properties of the topology τ shed light on the inner structure of the above-mentioned C^*-algebras. We formulate the results on sufficient conditions for non-triviality of the centers of the inductive limit $\mathfrak{B}^{K_i}$ and the direct product $\widehat{\mathfrak{M}}_{\mathcal{F}}$.

Theorem 4. [3, Theorem 3] *Let $i \in I$ be a non-isolated point with a countable neighbourhood base. Then the C^*-algebra $\mathfrak{B}^{K_i}$ has a non-trivial center.*

As a consequence of Theorem 4 and the definition of the C^*-algebra $\widehat{\mathfrak{M}}_{\mathcal{F}}$, we have the last statement in this section.

Corollary 5. *Let (I, τ) be a first-countable topological space without isolated points. Then the C^*-algebra $\widehat{\mathfrak{M}}_{\mathcal{F}}$ has a non-trivial center.*

6. INDUCTIVE SYSTEMS OF TOEPLITZ ALGEBRAS OVER ARBITRARY POSETS

In this section we specify the results of previous sections to the case of the inductive systems over partially ordered sets consisting of Toeplitz algebras.

Let K be an arbitrary partially ordered set. As above, we consider K as union (3) of its all maximal upward directed subsets $K_i, i \in I$ from Lemma 1.

We consider an inductive system

$$\mathcal{F} = (K, \{\mathcal{T}_a\}, \{\mathrm{id}_{ba}\}) \tag{25}$$

consisting of Toeplitz algebras, that is, $\mathcal{T}_a = \mathcal{T}$ for all $a \in K$, and the bonding $*$-homomorphisms $\mathrm{i}_{ba} : \mathcal{T}_a \longrightarrow \mathcal{T}_b$ are the identity mappings whenever $a, b \in K$ and $a \leq b$.

For each index $i \in I$ we consider an inductive subsystem $\mathcal{F}_i = (K_i, \{\mathcal{T}_a\}, \{\mathrm{id}_{ba}\})$ of system (25) over the maximal directed set K_i .

Let us construct inductive limits (6) of these inductive subsystems:

$$(\mathcal{T}^{K_i}, \{\mathrm{id}_a^{K_i}\}) := \varinjlim \mathcal{F}_i = \varinjlim(K_i, \{\mathcal{T}_a\}, \{\mathrm{id}_{ba}\}).$$

It is clear that one has the isomorphism $\mathcal{T}^{K_i} \simeq \mathcal{T}$ in the category of C^*-algebras.

Further, we take any element $a \in K$ and consider direct product (20) of C^*-algebras

$$\mathfrak{B}_a := \prod_{i \in U_a} \mathcal{T}^{K_i}.$$

Again, for every pair of elements $a, b \in K$ such that the inclusion $U_b \subset U_a$ holds, we define the $*$-homomorphism $\tau_{ba} : \mathfrak{B}_a \longrightarrow \mathfrak{B}_b$ by formula (21). Then we construct inductive system (22) and, for each index $i \in I$, we consider the inductive subsystem $(K_i, \{\mathfrak{B}_a\}, \{\tau_{ba}\})$ and its inductive limit (23), that is, the C^*-algebra $\mathfrak{B}^{K_i}$.

The following auxiliary result is valid. Its proof is based on the universal property for the inductive limits in the category of C^*-algebras.

Lemma 6. *Let $i \in I$ be a non-isolated point with a countable neighbourhood base*

$$\{U_{a_n} | a_n \in K_i, n \in \mathbb{N}\}$$

satisfying the condition $U_{a_1} \supset U_{a_2} \supset U_{a_3} \supset \ldots$. Let

$$(\mathfrak{B}, \{\tau_n\}) := \varinjlim(\{\mathfrak{B}_{a_n}\}, \{\tau_{a_{n+1}a_n}\})$$

be the inductive limit of the inductive sequence

$$\mathfrak{B}_{a_1} \xrightarrow{\tau_{a_2 a_1}} \mathfrak{B}_{a_2} \xrightarrow{\tau_{a_3 a_2}} \mathfrak{B}_{a_3} \xrightarrow{\tau_{a_4 a_3}} \ldots,$$

where $\tau_{a_{n+1}a_n}(f)(j) = f(j)$ for $f \in \mathfrak{B}_{a_n}$ and $j \in U_{a_{n+1}}$. Then there exists an isomorphism of C^*-algebras

$$\mathfrak{B} \simeq \mathfrak{B}^{K_i}.$$

Using Lemma 6, Coburn's theorem [16, theorem 3.5.18], [1, Proposition 1] and [33, Appendix L, Lemma L.1.3], we obtain

Theorem 5. [5, Theorem 1] *Let $i \in I$ be a non-isolated point with a countable neighbourhood base. Then for every sequence of natural numbers $M = (m_1, m_2, m_3, \ldots)$ there exists an injective $*$-homomorphism of C^*-algebras:*

$$C_r^*(\mathbb{Q}_M^+) \longrightarrow \mathfrak{B}^{K_i}.$$

Let us consider a sequence of all natural numbers $M = \mathbb{N}$. It is straightforward to check that the following equality holds for semigroups of rational numbers:

$$\mathbb{Q}_M^+ = \mathbb{Q}^+ := \mathbb{Q} \cap [0, +\infty).$$

As a consequence of Theorem 5, we immediately have

Theorem 6. [5, Theorem 2] *Let $i \in I$ be a non-isolated point with a countable neighbourhood base. There exists an injective $*$-homomorphism of C^*-algebras:*

$$C_r^*(\mathbb{Q}^+) \longrightarrow \mathfrak{B}^{K_i}.$$

Finally, we formulate the result that follows from Theorem 1.

Theorem 7. [4 Theorem 2] *Let K be a partially ordered set and let $\{K_i | i \in I\}$ be a family of all maximal directed subsets in the set K. Let $\mathcal{F}$ be an inductive system of Toeplitz algebras over K defined by a set of natural numbers N satisfying factorization equalities. Let $\mathcal{F}_i$, where $i \in I$, denote the inductive subsystem of $\mathcal{F}$ over the set K_i. Then there exists a family $\left\{\mathbb{Q}_{M_i}^+ | i \in I\right\}$ of semigroups of non-negative rational numbers and an isomorphism between the direct products of C^*-algebras*

$$\prod_{i \in I} \mathcal{T}^{K_i} \simeq \prod_{i \in I} C_r^*(\mathbb{Q}_{M_i}^+)$$

in the category of unital C^*-algebras and their unital $*$-homomorphisms.

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